ON DISCONTINUITY OF PLANAR OPTIMAL TRANSPORT MAPS

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ABSTRACT. Consider two bounded domains Ω and Λ in \mathbb{R}^2 , and two sufficiently regular probability measures μ and ν supported on them. By Brenier's theorem, there exists a unique transportation map T satisfying $T_{\#}\mu = \nu$ and minimizing the quadratic cost $\int_{\mathbb{R}^n} |T(x) - x|^2 d\mu(x)$. Furthermore, by Caffarelli's regularity theory for the real Monge–Ampère equations, if Λ is convex, T is continuous.

We study the reverse problem, namely, when is T discontinuous if Λ fails to be convex? We prove a result guaranteeing the discontinuity of T in terms of the geometries of Λ and Ω in the two-dimensional case. The main idea is to use tools of convex analysis and the extrinsic geometry of $\partial \Lambda$ to distinguish between Brenier and Alexandrov weak solutions of the Monge–Ampère equation. We also use this approach to give a new proof of a result due to Wolfson and Urbas.

We conclude by revisiting an example of Caffarelli, giving a detailed study of a discontinuous map between two explicit domains, and determining precisely where the discontinuities occur.

1. INTRODUCTION

Much work has gone into finding sufficient conditions for the optimal transportation map (OTM) to be continuous. According to Caffarelli [1], the OTM between two smooth densities (uniformly bounded away from zero and infinity) defined on bounded domains in \mathbb{R}^n with smooth boundaries is continuous when the target domain is convex. When n = 2, Figalli [4, Theorem 3.1] showed that even when the target domain is not convex, the OTM is still continuous outside a set of measure zero. This result has subsequently been extended to n > 2 by Figalli and Kim [5]. These results have also been studied for Riemannian manifolds, see the recent survey by De Philippis and Figalli [2]. However, there seems to be no known condition guaranteeing the discontinuity of planar OTMs. The main result of this article is such a condition.

In the present article we restrict ourselves to n = 2. Throughout this article, we denote the uniform probability measures on Ω and Λ by

$$\mu := rac{1}{|\Omega|} 1_\Omega \quad ext{and} \quad
u := rac{1}{|\Lambda|} 1_\Lambda,$$

respectively. We suppose for simplicity that Ω and Λ have unit area, i.e., $|\Omega| = |\Lambda| = 1$.

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Of course, one could consider more general probability measures, and certainly the results we discuss below carry over to measures with smooth densities that are uniformly bounded away from zero and from above. Our main interest is in the following:

Problem 1. Give conditions on Ω and Λ guaranteeing the discontinuity of the optimal transportation map from μ to ν .

The main result of this note is a sufficient condition guaranteeing the discontinuity of the OTM between two domains in \mathbb{R}^2 assuming the source domain Ω is convex. This condition can be phrased solely in terms of the geodesic curvature of the boundary of the target domain. Moreover, we give examples to show that the numerical constant in our condition is essentially sharp. Nevertheless, we show that the condition is not a necessary one for discontinuity. In addition, we give an alternative proof of a result of Wolfson and Urbas on the nonexistence of an OTM that extends smoothly to the boundary between arbitrary domains in \mathbb{R}^2 . Our methods are different from theirs in that we rely on cyclical monotonicity. This is what allows us to prove interior discontinuity as opposed to just non-smoothness up to the boundary. Finally, we revisit an example of Caffarelli and analyze precisely where the discontinuities occur using symmetry arguments and results of Caffarelli and Figalli. Unlike Caffarelli, we give a constructive proof of the discontinuity, and quantify where and how this discontinuity appears.

This note is organized as follows: In Section 2, we state and prove our conditions for discontinuity. We also give several examples illustrating when these conditions do and do not hold. Then, in Section 3, we consider a concrete example (which we term the "squareman") of an optimal map between two domains in which we can precisely determine how the map fails to be continuous. Finally, in Appendix A, we state and prove several lemmas that we need for the proof of the curvature condition.

2. A sufficient condition for discontinuity

In this section we derive a sufficient condition for the discontinuity of the OTM between Ω and Λ based on the geometry of the boundaries. We further show how our method proves a result of Wolfson, which was subsequently refined by Urbas. It is interesting to note that while Wolfson's original proof uses symplectic geometry, our approach is based on convex analysis.

Consider a simple closed C^2 curve $C \subset \mathbb{R}^2$, and let **n** denote the inward-pointing unit normal along C. Given a unit-speed parametrization $\gamma : I \to \mathbb{R}^2$ of C (here, $I \subset \mathbb{R}$ denotes an interval, which we can assume equals [0, L] without any loss of generality), the curvature of C is defined to be the function $\kappa : C \to \mathbb{R}$ satisfying $\gamma'' = \kappa \mathbf{n}$. Note that since we have defined the signed curvature with respect to the inward pointing unit normal, it is independent of the orientation of the curve.

In this article, we will refer numerous times to connected subsets or connected components of a simple (possibly closed) curve. Both of these simply refer to a subset of the (image of the) curve which is connected (and hence, path connected) in the subspace topology induced from \mathbb{R}^2 . In particular, we are not referring to maximally (path) connected components of the curve. Since a continuous bijection from a compact space to a Hausdorff space is automatically a homeomorphism, and all our curves have domain

[0,1] or S^1 , it follows that a subset of the curve γ is connected if and only if it is of the form $\gamma(I)$, where I is a sub-interval of [0, 1] or S^1 .

Theorem 2. Let Ω and Λ be bounded, connected, simply connected open domains in \mathbb{R}^2 such that $\partial\Omega$ and $\partial\Lambda$ are C^3 , closed curves. Assume Ω is convex. Equip Ω and Λ with the uniform measures μ and ν . Let $\kappa_{\partial\Omega}$ and $\kappa_{\partial\Lambda}$ be the signed curvatures of $\partial\Omega$ and $\partial \Lambda$ with respect to the corresponding inward-pointing unit normal fields. If there exists a connected subset $J \subset \partial \Lambda$ with

(1)
$$\int_J \kappa_{\partial \Lambda} < -\pi,$$

then T_1 , the OTM from Ω to Λ , is discontinuous.

We employ similar techniques, together with an additional modification, to give a new proof of the following result due to Wolfson and Urbas [10, 7].

Theorem 3 (Wolfson and Urbas). Let Ω and Λ be two bounded, connected, simply connected domains in \mathbb{R}^2 with C^2 boundaries. Let $\kappa_{\partial\Omega}$ and $\kappa_{\partial\Lambda}$ denote the signed curvatures (as defined above) of the two boundaries. Assume that

(2)
$$\inf_{J\subset\partial\Lambda}\int_{J}\kappa_{\partial\Lambda}\leq\inf_{I\subset\partial\Omega}\int_{I}\kappa_{\partial\Omega}-\pi,$$

where I and J are connected subsets of $\partial\Omega$ and $\partial\Lambda$, respectively. Then there does not exist a C^1 -diffeomorphism $T_1: \overline{\Omega} \to \overline{\Lambda}$ whose restriction to Ω is an OTM.

When Ω is convex, of course $\kappa_{\partial\Omega} \geq 0$. One may ask whether (1) may be weakened. Below, we will construct an example (Example 6) to show that at least when the C^3 hypothesis in Theorem 2 is replaced with piecewise smooth, the constant $-\pi$ in (1) cannot be increased. However, (1) is not a necessary condition: in Section 3 we will construct an example where Ω is convex and Λ has a connected subset of total curvature of at most $-\frac{\pi}{2}$, but $OTM(\Omega, \Lambda)$ is discontinuous.

Condition (2) is also not necessary: below (Example 5), we construct Ω and Λ so that $\inf_{I \subset \partial \Lambda} \int_{I} \kappa_{\partial \Lambda} = \inf_{J \subset \partial \Omega} \int_{I} \kappa_{\partial \Omega}$, but $OTM(\Omega, \Lambda)$ is not a C^1 diffeomorphism up to the boundary.

The strength of Theorem 2 lies in the fact that it does not assume any nice behavior of the optimal map near the boundary. At the same time it shows not only lack of regularity, but discontinuity. In the proof of Theorem 2, the convexity is used to show a continuous OTM from Ω to Λ is necessarily a C^1 -diffeomorphism, by Caffarelli's regularity theorem. If we are concerned only with the nonexistence of OTMs which are C^1 diffeomorphisms up to the boundary, one can do away with the convexity assumption, which is the content of Theorem 3.

The proof of Theorem 3 contains two differences from the proof of Theorem 2. First, the technical Lemma 4 is no longer necessary, since we are assuming regularity up to the boundary. On the other hand, condition (2) is weaker than (1), and so one must make use of cyclical monotonicity and not just of monotonicity.

Proof of Theorem 2. Assume for the sake of contradiction that the OTM

 $T_1: \Omega \to \Lambda$

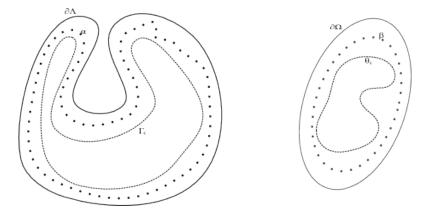


FIGURE 1. $\alpha = T_1(\beta)$ is "close" to $\partial \Lambda$ and β is convex.

is continuous everywhere on Ω . Since Ω is convex, we have from Caffarelli's regularity theorem [9, Theorem 12.50] that the map T_2 (the optimal map from Λ to Ω) is C^2 everywhere on Λ . We also know that for μ -almost all x and for ν -almost all $y, T_2 \circ$ $T_1(x) = x$ and $T_1 \circ T_2(y) = y$ [8, Theorem 2.12]. Since both compositions are continuous and are equal to the identity almost everywhere, it follows that they must be the identity everywhere, and therefore, that $T_2^{-1} = T_1$ everywhere. Moreover, since T_2 is C^2 by Caffarelli's regularity theorem, and since its Jacobian matrix is nonsingular at every point in Λ by the Monge-Ampère equation, it follows from the inverse function theorem that T_1 is also C^2 . Hence T_2 is a C^2 -diffeomorphism between Λ and Ω .

For sufficiently small $\epsilon > 0$, consider the sets (see Figure 1)

$$\Lambda_{\epsilon} = \{ x \in \Lambda : \operatorname{dist}(x, \partial \Lambda) < \epsilon \}, \qquad \Gamma_{\epsilon} = \{ x \in \Lambda : \operatorname{dist}(x, \partial \Lambda) = \epsilon \}.$$

From Proposition 8, we know that there exists $\hat{\epsilon} > 0$ such that for every $0 < \epsilon \leq \hat{\epsilon}$, the curve Γ_{ϵ} is C^1 . Furthermore, there exists a diffeomorphism $f_{\epsilon} : \partial \Lambda \to \Gamma_{\epsilon}$ such that the vector $f_{\epsilon}(x) - x$ is normal to $\partial \Lambda$ at x and to Γ_{ϵ} at $f_{\epsilon}(x)$, and has magnitude $|f_{\epsilon}(x) - x| = \epsilon$. Consider the image $T_2(\Gamma_{\epsilon}) = \Theta_{\epsilon}$.

Since T_2 is C^2 and Γ_{ϵ} is compact, we have that $\Theta_{\epsilon} \subset \Omega$ is also a compact, closed C^1 curve, so that in particular, $\operatorname{dist}(\partial\Omega, \Theta_{\epsilon}) > \delta > 0$. In order to be able to work in the interior of Ω , we would like to construct a C^2 convex curve $\beta \subset \Omega$ such that the interior of the region enclosed by β completely contains Θ_{ϵ} . Note that here, and elsewhere, we use the standard terminology of calling a closed curve convex if it is the boundary of a bounded convex set. Such a β can readily be constructed: we pick a point x_0 contained in the interior of the region bounded by Θ_{ϵ} and scale points on $\partial\Omega$ with respect to x_0 by a factor $1 - \tilde{\epsilon} < t < 1$ for a small enough $\tilde{\epsilon} > 0$. Denote the resulting curve by β_t . Then, β_t is seen to be convex and C^2 , since $\partial\Omega$ is convex and C^2 (in fact we have assumed that it is C^3). Further, we can always arrange $\operatorname{dist}(\partial\Omega, \beta_t) < \frac{\delta}{2}$ by picking $\tilde{\epsilon} > 0$ small enough. In particular, we can choose t so that $\beta = \beta_t$ contains Θ_{ϵ} completely in its interior, and is as close to $\partial\Omega$ as desired. Note that the convexity of β implies that the signed curvature κ_{β} of β with respect to the inward unit normal field is non-negative.

Next, consider the C^2 curve $\alpha = T_1(\beta)$. We claim that α contains Γ_{ϵ} in its interior in the sense that every continuous path between Γ_{ϵ} and $\partial \Lambda$ must intersect α . Indeed, let

 $c: [0,1] \to \Lambda$ (note that we may assume without loss of generality that $c([0,1)) \subset \Lambda$) be a continuous map such that $c(0) \in \Gamma_{\epsilon}$ and $c(1) \in \partial \Lambda$. Then, $T_2(c(0)) \in \Theta_{\epsilon}$, while $\cap_{t \in (0,1)} \overline{T_2(c(t,1))} \neq \emptyset$ by the finite intersection property applied to the compact space $\overline{\Omega}$. We claim that $\cap_{t \in (0,1)} T_2(c(t,1)) \subset \partial \Omega$. Indeed, suppose that $\cap_{t \in (0,1)} T_2(c(t,1)) \nsubseteq \partial \Omega$. Since the intersection is nonempty and contained in $\overline{\Omega}$, we must have some point $q \in \Omega$ such that $q \in \bigcap_{t \in (0,1)} T_2(c(t,1))$. Consider the point $p = T_1(q)$. Since $p \in \Lambda$, we can (by the continuity of c) find some $t' \in (0, 1)$ sufficiently close to 1 such that $p \notin c[t', 1]$. Since T_2 is an open map on Λ by hypothesis, it follows that $q = T_2(p) \notin \overline{T_2(c(t', 1))}$, which gives us a contradiction. Once we have that $\bigcap_{t \in (0,1)} \overline{T_2(c(t,1))} \subset \partial \Omega$, we can prove that $T_2(c)$ intersects β , and hence, that c intersects α . For this, it clearly suffices to show that there exists some $t_0 \in (0,1)$ such that for all $t \in (t_0,1)$ one has $dist(T_2(c(t)),\partial\Omega) < 0$ dist($\beta, \partial \Omega$). Suppose such a t_0 does not exist. Then, there exists a sequence $t_n \uparrow 1$ such that $p_n = T_2(c(t_n))$ sits inside the closure of the domain bounded by β (we will denote this domain by dom(β)). By compactness of dom(β), we can assume after possibly passing to a subsequence that $p_n \to p$ in dom (β) . But then, we have that $p \in \Omega$, and $p \in \bigcap_{t \in (0,1)} T_2(c(t,1))$, which is a contradiction. Note that since connected components are preserved under homeomorphisms, Γ_{ϵ} is a Jordan curve, and α contains at least one point in Γ_{ϵ} , we have also showed that $\alpha \subset \Gamma_{\epsilon}$.

We will need the following lemma.

Lemma 4. Let κ_{α} be the signed curvature of α with respect to the inward pointing unit normal vector field. Then for sufficiently small ϵ there exists some connected subset $I_1 \subset \alpha$ for which the total signed curvature is less than $-\pi$, i.e., $\int_{I_1} \kappa_{\alpha} < -\pi$.

The lemma says that a closed curve "close" to the boundary of our domain must exhibit similar curvature behaviour. We will prove this after we show how it implies the curvature condition.

Using Lemma 4, we can pick a connected $I_1 \subset \alpha$ such that $\int_{I_1} \kappa_\alpha < -\pi$. Let $|I_1| = l > 0$ be the length of I_1 . We denote a unit speed parametrization for I_1 by $\alpha_1 : [0, l] \to I_1$. Since T_2 is an optimal map from ν to μ , we have from monotonicity [8, Proposition 2.24] that for any two points $x, y \in \alpha$, $\langle x - y, T_2(x) - T_2(y) \rangle \geq 0$. In particular, for every $t \in [0, l)$ and for a suitably small h > 0, we must have that

$$\langle \alpha_1(t+h) - \alpha_1(t), T_2(\alpha_1(t+h)) - T_2(\alpha_1(t)) \rangle \ge 0$$

Dividing by h^2 and letting $h \to 0$ in the previous equation, we get that

(3)
$$\langle \dot{\alpha_1}(t), \dot{\beta_1}(t) \rangle \ge 0$$

where $\beta_1 = T_2 \circ \alpha_1$ is a parametrization of $T_2(I_1)$. Note $\dot{\beta}_1(t) \neq 0$ as T_2 is locally a diffeomorphism around every $\alpha_1(t) \in \Lambda$.

Equation (3) implies that the tangent vectors to α and β at any $x \in \alpha$ and $T_2(x) \in \beta$ must have a non-negative inner product. We show that this cannot happen, thereby proving Theorem 2 (modulo the proof of Lemma 4). Intuitively, it is clear that this cannot happen: as we move along I_1 counterclockwise, the tangent vector at a point along I_1 rotates clockwise, while the tangent vector at the corresponding point on β rotates counterclockwise. Monotonicity dictates that the angle between the corresponding vectors must always be within $\frac{\pi}{2}$. However, since the curvature of I_1 is less than $-\pi$, and the curvature of the corresponding connected subset of β is ≥ 0 by convexity, the angle between corresponding tangent vectors changes by more than $-\pi$ when traversing I_1 , and therefore, cannot lie in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ at all points of I_1 .

To make the above discussion more precise, consider the "tail-to-tail" angle between two vectors. This is the standard notion of angle which takes values in the interval $(-\pi,\pi]$. Given an ordered pair of vectors, we define the angle between them as the signed "tail-to-tail angle" between them, with the sign taken to be positive if we move counterclockwise from the first vector to the second and negative otherwise. We now define $f: [0, l] \to (-\pi, \pi]$, where f(t) is the angle from $\dot{\beta}_1(t)$ to $\dot{\alpha}_1(t)$. From (3) it follows that $f(t) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ for every $t \in [0, l]$. Set $J_t = \alpha_1([0, t]) \subset I_1$. Then

(4)
$$f(t) - f(0) = \int_{J_t} \kappa_\alpha - \int_{T_2(J_t)} \kappa_\beta + 2k(t)\pi$$

where $k(t) \in \mathbb{Z}$.

Since α and β are C^2 , the unsigned angle between $\dot{\alpha}_1(t)$ and $\dot{\beta}_1(t)$ defined from [0, l] to $[0, \infty)$ varies continuously. Therefore, any discontinuities in the signed angle f(t) can occur only near the values $-\pi$ and π . But since $f(t) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, it is never close to π or $-\pi$ and therefore must be continuous everywhere on [0, l]. Hence $2k(t)\pi$ must also be continuous. But k(t) is integer-valued, so it is the constant k(0) = 0. In particular (4) yields

(5)
$$f(t) = \int_{J_t} \kappa_{\alpha} - \int_{T_2(J_t)} \kappa_{\beta} + f(0).$$

Setting t = l in (5), we get

$$f(l) - f(0) = \int_{J_l} \kappa_{\alpha} - \int_{T_2(J_l)} \kappa_{\beta} = \int_{I_1} \kappa_{\alpha} - \int_{T_2(I_1)} \kappa_{\beta} < -\pi + 0 = -\pi,$$

where the inequality holds since $\int_{I_1} \kappa_{\alpha}(x) dx < -\pi$ and $\int_J \kappa_{\beta}(x) dx \ge 0$ for every $J \subset \beta$ as β is the boundary of a convex set. On the other hand $f(t) - f(0) \in [-\pi, \pi]$ since $f(t) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ for all $t \in [0, l]$. This contradicts $f(l) - f(0) < -\pi$. Hence our assumption that T_1 is continuous must be incorrect, and T_1 is discontinuous as desired. \Box

Proof of Lemma 4. Recall $\Gamma_{\epsilon} = \{x \in \Lambda : \operatorname{dist}(x, \partial \Lambda) = \epsilon\}$, and α contains Γ_{ϵ} in its interior in the sense that every continuous path between Γ_{ϵ} and $\partial \Lambda$ must intersect α . Also recall that by assumption, $\int_{I} \kappa_{\partial \Lambda} < -\pi$ for some connected subset $I \subset \partial \Lambda$. The underlying idea is to choose a connected subset of α which is close to the connected subset I of $\partial \Lambda$, and then show that this subset must necessarily contain a further connected subset of signed curvature less than $-\pi$. We have illustrated the arguments made below in Figure 2.

Let A and B be the endpoints of I. Let $\overline{\epsilon} > 0$. Then pick $C \in I$ close to A so that the angle between AC and the tangent to $\partial \Lambda$ at A is less than $\frac{\overline{\epsilon}}{2}$. Similarly pick some $D \in I$ close to B which satisfies the same criterion. For $\epsilon > 0$ small, consider the curve Γ_{ϵ} . Recall $f_{\epsilon} : \Gamma \to \Gamma_{\epsilon}$ maps points on Γ to points ϵ away on Γ_{ϵ} . Now $A_2 = f_{\epsilon}(A)$, $B_2 = f_{\epsilon}(B), C_2 = f_{\epsilon}(C)$ and $D_2 = f_{\epsilon}(D)$ are points on Γ_{ϵ} . By selecting $\epsilon > 0$ small enough, we can ensure that for any $X \in AA_2$ and any $Y \in CC_2$, the angle between AA_2 and XY belongs to the interval $(\frac{\pi}{2} - \overline{\epsilon}, \frac{\pi}{2} + \overline{\epsilon})$, and also that for any $X_1 \in BB_2$

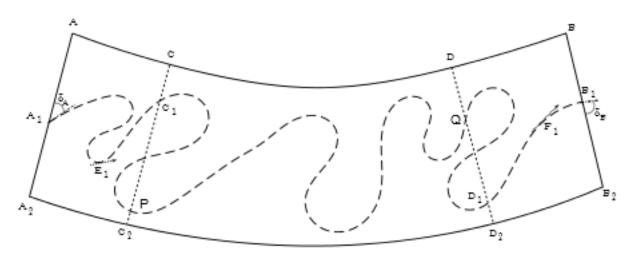


FIGURE 2. The curve $A_1B_1 \subset \alpha$ enclosed by the region of negative curvature.

and $Y_1 \in DD_2$, the angle between BB_2 and X_1Y_1 belongs to the interval $(\frac{\pi}{2} - \bar{\epsilon}, \frac{\pi}{2} + \bar{\epsilon})$. This is equivalent to saying that the direction of XY differs by no more than $\bar{\epsilon}$ from the direction of $\partial \Lambda$ at the point A, and the direction of X_1Y_1 differs by no more than $\bar{\epsilon}$ from the direction of $\partial \Lambda$ at the point B.

Next, from Proposition 10, we have that there exists a connected subset $I \subset \alpha$ which is contained in the region bounded by I, AA_2 , BB_2 and Γ_{ϵ} , and which has endpoints $A_1 \in AA_2$ and $B_1 \in BB_2$. We move along \tilde{I} from A_1 to B_1 and denote by C_1 the point where \tilde{I} intersects CC_2 for the first time. We denote by D_1 the point where \tilde{I} intersects DD_2 for the last time. From Proposition 9, we further know that there exists a point E_1 lying on the portion of \tilde{I} between A_1 and C_1 at which the direction of the tangent to \tilde{I} coincides with the direction of A_1C_1 . In particular, the angle between the tangent to \tilde{I} at E_1 and the segment AA_2 is in the interval $(\frac{\pi}{2} - \bar{\epsilon}, \frac{\pi}{2} + \bar{\epsilon})$. Similarly, we can choose a point F_1 that lies on the portion of \tilde{I} connecting D_1 to B_1 such that the angle between the tangent to \tilde{I} at F_1 and BB_2 is in the interval $(\frac{\pi}{2} - \bar{\epsilon}, \frac{\pi}{2} + \bar{\epsilon})$.

Finally, we are in a position to establish the existence of the interval $I_1 \subset \tilde{I}$ for which $\int_{I_1} \kappa_{\alpha} < -\pi$ with respect to the unit normal pointing inside the bounded component of the complement of the Jordan curve α . Equivalently, it is the signed curvature with respect to the standard orientation on \mathbb{R}^2 and with I_1 oriented from B_1 to A_1 . We will always use this sign of curvature for (connected components) of \tilde{I} . Denote by δ_A the angle between the tangent to \tilde{I} at the point A_1 and the vector A_1A and by δ_B , the angle between the vector B_1B_2 and the tangent to \tilde{I} . In this definition, we have used the tangent vector to \tilde{I} when it is oriented from A_1 to B_1 . Note that both δ_A and δ_B are in $[0,\pi]$. We now apply the Gauss-Bonnet theorem to the region (with its boundary oriented counterclockwise) bounded by I, \tilde{I} , AA_1 and BB_1 to get that

(6)
$$\int_{I} \kappa_{\partial \Lambda} + \frac{\pi}{2} + (\pi - \delta_A) - \int_{\tilde{I}} \kappa_{\alpha} + (\pi - \delta_B) + \frac{\pi}{2} = 2\pi.$$

Note the negative sign in front of the integral over I, which comes from the fact that the orientation of \tilde{I} in this calculation is from A_1 to B_1 , which is the opposite of what we had originally used (i.e. from B_1 to A_1) in computing $\int_{\tilde{I}} \kappa_{\alpha}$. Simplifying (6), we have $\int_{\tilde{I}} \kappa_{\alpha} = \int_{I} \kappa_{\partial\Lambda} + \pi - \delta_B - \delta_A$ We will now split \tilde{I} into three parts and show that some connected combination of these three parts has a total signed curvature lesser than $-\pi$ with the original orientation i.e. with \tilde{I} going from B_1 to A_1 . Note that the points E_1 and F_1 provide such a splitting naturally. Denote these three components of \tilde{I} between A_1 and E_1 , E_1 and F_1 , and F_1 and B_1 by \tilde{I}_1 , \tilde{I}_2 , \tilde{I}_3 respectively. For some $\hat{\epsilon}_1, \hat{\epsilon}_3 \in (-\bar{\epsilon}, \bar{\epsilon})$,

$$\int_{\tilde{I}_1} \kappa_\alpha = \left(\frac{\pi}{2} + \hat{\epsilon}_1\right) - \delta_A + 2k\pi$$

for some integer k, and

$$\int_{\tilde{I}_3} \kappa_\alpha = (\pi - \delta_B) - \left(\frac{\pi}{2} - \hat{\epsilon}_3\right) + 2k'\pi = \frac{\pi}{2} - \delta_B + 2k'\pi + \hat{\epsilon}_3$$

for some integer k'.

If k < 0, then

$$\int_{\tilde{I}_1} \kappa_\alpha < -\frac{3\pi}{2} + \overline{\epsilon}$$

Taking $\overline{\epsilon}$ small enough, we get

$$\int_{\tilde{I}_1} \kappa_\alpha < -\pi$$

in which case I_1 is the desired segment.

If k > 0, then

$$\int_{\tilde{I}\setminus\tilde{I}_1} \kappa_{\alpha} = \int_{I} \kappa_{\partial\Lambda} + \pi - \delta_B - \delta_A - \int_{\tilde{I}_1} \kappa_{\alpha}$$
$$< -\pi + \pi - \delta_B - \delta_A - \frac{\pi}{2} + \delta_A - 2\pi + \bar{\epsilon}$$
$$= -\delta_B - \frac{5}{2}\pi + \bar{\epsilon} < -\pi,$$

as long as we take $\bar{\epsilon} > 0$ sufficiently small. Hence, in this case we have that $\tilde{I} \setminus \tilde{I}_1 = \tilde{I}_2 \cup \tilde{I}_3$ has total curvature less than $-\pi$.

Similarly, if $k' \neq 0$, then using one of the arguments above, we can take I_1 to be either \tilde{I}_3 or $\tilde{I}_1 \cup \tilde{I}_2$.

Thus, the only case left to investigate is when k = k' = 0. In this case,

$$\int_{\tilde{I}_1} \kappa_{\alpha} = \frac{\pi}{2} - \delta_A + \hat{\epsilon}_1 \quad \text{and} \quad \int_{\tilde{I}_3} \kappa_{\alpha} = \frac{\pi}{2} - \delta_B + \hat{\epsilon}_3$$

Combining these two equations, we get that

$$\int_{\tilde{I}_2} \kappa_{\alpha} = \int_{\tilde{I}} \kappa_{\alpha} - \int_{\tilde{I}_1} \kappa_{\alpha} - \int_{\tilde{I}_3} \kappa_{\alpha}$$
$$= \int_{I} \kappa_{\partial\Lambda} + \pi - \delta_B - \delta_A - \frac{\pi}{2} + \delta_A - \frac{\pi}{2} + \delta_B - \hat{\epsilon}_1 - \hat{\epsilon}_3$$
$$< \int_{I} \kappa_{\partial\Lambda} + 2\bar{\epsilon}.$$

In particular, if we choose $\overline{\epsilon} < -\frac{\int_{I} \kappa_{\partial \Lambda} + \pi}{2}$, then, it follows that $\int_{\tilde{I}_{2}} \kappa_{\alpha} \leq \int_{I} \kappa_{\partial \Lambda} + 2\overline{\epsilon} < -\pi$ and thus in this case, \tilde{I}_{2} satisfies the claim. This completes the proof of the lemma. \Box

Proof of Theorem 3. Assume the existence of such T_1 . We follow the notation established in the proof of Theorem 2. Pick $J \subset \partial \Lambda$ so that $\int_J \kappa_{\partial \Lambda} \leq \inf_{I \subset \partial \Omega} \int_I \kappa_{\partial \Omega} - \pi$. We can choose such a $J \subset \partial \Lambda$ because by assumption, $\inf_{J \subset \partial \Lambda} \int_J \kappa_{\partial \Lambda} \leq \inf_{I \subset \partial \Omega} \int_I \kappa_{\partial \Omega} - \pi$, and the infimum on the left hand side is attained because $\partial \Lambda$ is compact. Let γ : $[0, l] \to J$ be the unit speed parametrization of J, so that $\gamma''(t) = \kappa_{\partial \Lambda}(\gamma(t))\mathbf{n}_{\partial \Lambda}(\gamma(t))$ where $\mathbf{n}_{\partial \Lambda}(\gamma(t))$ is the inward pointing unit normal at $\gamma(t)$ and l is the length of J. From monotonicity (recall (3)), $f(t) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ for all $0 \leq t \leq l$. But we also have that for every $t \in [0, l]$,

$$f(t) - f(0) = \int_{\gamma([0,t])} \kappa_{\partial \Lambda} - \int_{T_1^{-1}(\gamma([0,t]))} \kappa_{\partial \Omega} + 2k(t)\pi$$

where as before k(t) is an integer that accounts for the discontinuity that amounts from restricting the codomain of f to $(-\pi, \pi]$. Since $f(t) \in [-\frac{\pi}{2}, \frac{\pi}{2}], k = 0$. Thus

$$\int_{\gamma([0,t])} \kappa_{\partial \Lambda} - \int_{T_1^{-1}(\gamma([0,t]))} \kappa_{\partial \Omega} = f(t) - f(0).$$

On the other hand, from the choice of $J \subset \partial \Lambda$, we have that

$$\int_{T_1^{-1}(\gamma([0,l]))} \kappa_{\partial\Omega} - \int_{\gamma([0,l])} \kappa_{\partial\Lambda} = \int_{T_1^{-1}(J)} \kappa_{\partial\Omega} - \int_J \kappa_{\partial\Lambda} \ge \inf_{I \subset \partial\Omega} \int_I \kappa_{\partial\Omega} - \int_J \kappa_{\partial\Lambda} \ge \pi.$$

To conclude the proof, we claim that $f(t) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. To see this, observe that for any $x, y, z \in \overline{\Lambda}$

$$\langle x, T_2(x) - T_2(y) \rangle + \langle y, T_2(y) - T_2(z) \rangle + \langle z, T_2(z) - T_2(x) \rangle \ge 0,$$

by cyclical monotonicity, or equivalently

$$\langle z-y, T_2(z) - T_2(y) \rangle \ge \langle x-z, T_2(y) - T_2(x) \rangle.$$

Now by Brenier's theorem $T_2 = \nabla w$ for some convex function w on Λ which is C^2 by our assumptions, and since det $\nabla^2 w = 1$ this function is strongly convex in the sense that $\nabla^2 w > 0$. Since T_2 is C^1 we have $T_2(y) - T_2(x) = DT_2(x) \cdot (y - x) + v(y)$, where |v(y)| = o(|x - y|). Thus,

$$\langle z - y, T_2(z) - T_2(y) \rangle \ge \langle x - z, \nabla^2 w(x) \cdot (y - x) + v(y) \rangle$$

Now pick a closed disk that is contained in $\overline{\Lambda}$ s.t. the circle bounding the disk is tangent to $\partial \Lambda$ at x. Let $\tilde{\gamma}$ be a unit speed parametrization of this circle. In particular,

for some fixed t_2 we have that $\tilde{\gamma}(t_2) = x$. Set $y = \tilde{\gamma}(t_1)$ and $z = \tilde{\gamma}(t_3)$ and such that |y - x| = |z - x| or equivalently $2t_2 = t_1 + t_3$. Also, set $\delta(t) := T_2 \circ \tilde{\gamma}(t)$. Then,

$$\langle \tilde{\gamma}(t_3) - \tilde{\gamma}(t_1), \delta(t_3) - \delta(t_1) \rangle \ge \langle x - z, \nabla^2 w(x) \cdot (y - x) + v(y) \rangle$$

Note that for sufficiently small t_3-t_1 , there exists a constant C > 0 such that $\frac{1}{C}|z-y| < t_3 - t_1 < C|z-y|$ and $|x-y| \leq |z-y| \leq 2|x-y|$. Thus dividing both sides of the equation by $(t_3 - t_1)^2$ and taking the limit as $t_3 - t_1$ tends to zero gives

$$\langle \dot{\tilde{\gamma}}(t_2), \dot{\delta}(t_2) \rangle \ge C' \nabla^2 w \langle \nu, \nu \rangle > 0,$$

as $|\nu| = 1$ is the tangent vector to the disk at x and C' > 0. This completes the proof of the theorem.

2.1. **Examples.** In this section we will explore several concrete examples. The first one shows that the extended curvature criterion for the non-existence of OTMs which are diffeomorphisms up to the boundary is sufficient but not necessary. The second example shows that it is reasonable not to hope for a constant better than $-\pi$ in the curvature criterion.

Example 5. Consider the two domains pictured in Figure 3 and suppose there exists an optimal map $T: \overline{\Omega} \to \overline{\Lambda}$, which is a C^1 -diffeomorphism. Given $\epsilon > 0$ small, we construct our domains so that $-2\pi \leq \inf_{I \subset \partial \Omega} \int_I \kappa_{\partial \Omega} = \inf_{J \subset \partial \Lambda} \int_J \kappa_{\partial \Lambda} < -2\pi + \epsilon$. In particular, the hypotheses of Theorem 3 (and, of course, those of Theorem 2, since Ω is not convex) do not apply here.

Note that $\partial \Lambda$ has 4 disjoint connected subsets $\{J_1, \ldots, J_4\}$ of signed curvature $-2\pi + \tilde{\epsilon}$ for some $0 < \tilde{\epsilon} < \epsilon$, while $\partial \Omega$ has only one negatively curved component, with signed curvature no lesser than -2π . In particular, the integral of $\kappa_{\partial\Omega}$ over any union of connected subsets of $\partial\Omega$ cannot be lesser than -2π .

As before, we show that an OTM T (as above) cannot exist, by showing that it violates the monotonicity condition for OTMs. Indeed, by the "tail-to-tail" argument used in the proof of Theorem 2, we have that

$$-\pi \leq \int_{J_i} \kappa_{\partial \Lambda} - \int_{T(J_i)} \kappa_{\partial \Omega} < -2\pi + \tilde{\epsilon} - \int_{T(J_i)} \kappa_{\partial \Omega}$$

so that

$$\int_{T(J_i)} \kappa_{\partial\Omega} < -\pi + \tilde{\epsilon} < -\pi + \epsilon$$

Note that the images $T(J_i)$ are disjoint. Therefore, from the above discussion, we have

$$-2\pi \le \int_{\cup T(J_i)} \kappa_{\partial\Omega} = \sum_{i=1}^4 \int_{T(J_i)} \kappa_{\partial\Omega} < -4\pi + 4\epsilon < -2\pi$$

for ϵ small enough, which is a contradiction. This completes the proof of the nonoptimality of T.

Example 6. Let $\Omega = \{(x, y) \in \mathbb{R}^2 | x > 0, 1 < x^2 + y^2 < 2\}$ be a half annulus and $\Lambda = \{(x, y) \in \mathbb{R}^2 | x > 0, x^2 + y^2 < 1\}$ be the unit half disk. We equip them with the restricted Lebesgue measure. Note that this equips both Ω and Λ with probability measures. Further note that Λ is convex, while Ω has a segment of curvature (the inner

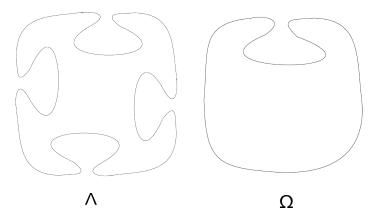


FIGURE 3. Both domains have boundary segments that are equally negatively curved, but the OTM is not a smooth diffeomorphism up to the boundary.

half circle) $-\pi$. Also observe that both boundaries are piecewise smooth. We will show that the OTM $T: \Omega \to \Lambda$ is continuous on the interior of Ω .

Indeed we expect the map to preserve the points radially and to "contract" the half annulus to the half disk by preserving the area. Such a map would take the form

$$T(x,y) = \left(x\sqrt{1 - \frac{1}{x^2 + y^2}}, y\sqrt{1 - \frac{1}{x^2 + y^2}}\right).$$

It is immediate to see T is a diffeomorphism between Ω and Λ with det DT = 1, so that T is area preserving. Further, $T = \nabla \varphi$ where $\varphi : \Omega \to \mathbb{R}$ is the smooth convex function given by

$$\varphi(x,y) = \frac{\sqrt{x^2 + y^2}\sqrt{x^2 + y^2 - 1} - \log\left(\sqrt{x^2 + y^2} + \sqrt{x^2 + y^2 - 1}\right)}{2}.$$

But now T is the gradient of a convex function, and is also area preserving. By Brenier's theorem [8, Theorem 2.12], it is the unique OTM between Ω and Λ .

3. The squareman

One of the characteristics of the subject of optimal transport is that despite many deep results on existence and regularity of OTMs, it is still very hard to explicitly compute the OTM in almost any non-trivial example. Caffarelli gave an example of a discontinuous OTM to show that without convexity of the target his regularity theory could break down [1] (see also [9, Theorem 12.3]). He showed that the OTM between a disk and two half disks connected via a sufficiently thin bridge is discontinuous. However, it is not exactly clear how "thin" the bridge should be and where and how the discontinuity arises. In this section we will consider an example very close to Caffarelli's example, and hopefully provide the reader with some intuition of what the map looks like. In particular, we will discuss where and how the discontinuity arises in this example, and make some qualitative statements about the extent of this discontinuity. Our computation relies on results due to Caffarelli and Figalli and we begin by recalling some of these results.

Recall that if μ and ν are two probability measures (not necessarily uniform) supported on Ω and Λ respectively, then we denote by T_1 the optimal map which transports μ to ν . Similarly, T_2 is the optimal map that transports ν to μ . In our setting, where μ, ν are sufficiently regular measures supported on domains in \mathbb{R}^2 and the transportation cost is the quadratic cost, we have that $T_2 = \nabla \phi$ for some convex function ϕ on Λ and $T_1 = \nabla \phi^*$ on Ω , where ϕ^* is the Legendre transform of ϕ [8, Theorem 2.12]. Let γ be the optimal transportation plan between μ and ν i.e. $\gamma = (Id \times T_1)_{\#}\mu$. We will need the following short lemma:

Lemma 7. Assume that μ and ν are uniform measures supported on $\tilde{\Omega}$ and $\tilde{\Lambda}$ respectively, where $\tilde{\Omega}$ and $\tilde{\Lambda}$ are bounded, connected, simply connected, open domains in \mathbb{R}^2 . Let $\tilde{\Omega}$ be convex. Then the OTM from $\tilde{\Lambda}$ to $\tilde{\Omega}$, denoted by T, is smooth. Further, T is a diffeomorphism between $\tilde{\Lambda}$ and an open set $\tilde{\Omega}' \subset \tilde{\Omega}$ of full measure.

Proof. Since $\tilde{\Omega}$ is convex, Caffarelli's regularity theory is applicable. Hence T is smooth on $\tilde{\Lambda}$. By the Monge–Ampère equation, $\det(DT) = \frac{|\tilde{\Omega}|}{|\tilde{\Lambda}|} = c$ for some c > 0 almost everywhere on $\tilde{\Lambda}$. Since T is smooth, it follows that $\det(DT) = c > 0$ everywhere. In particular, by the inverse function theorem T is a local diffeomorphism at every point of $\tilde{\Lambda}$. To show that T is a global diffeomorphism between $\tilde{\Lambda}$ and $T(\tilde{\Lambda})$, we only need to show that T is injective. This follows, for instance, from Caffarelli's result on strict convexity of solutions to the Monge–Ampère equation above, but we also give an elementary argument.

Indeed, assume on the contrary that there exist $x, y \in \tilde{\Lambda}$ such that T(x) = T(y) = z. Pick $\epsilon > 0$ such that $B_{\epsilon}(x) \cap B_{\epsilon}(y) = \emptyset$ and T is a diffeomorphism when restricted separately to both $B_{\epsilon}(x)$ and $B_{\epsilon}(y)$. Let $A = T(B_{\epsilon}(x)) \cap T(B_{\epsilon}(y))$. Since A is nonempty and open, it must have positive measure. But then, for every $a \in A$ the set $\{b \in \tilde{\Lambda} : (a, b) \in \operatorname{supp}(\gamma)\}$ contains at least two elements - one from $B_{\epsilon}(x)$ and one from $B_{\epsilon}(y)$, where γ is the optimal transportation plan between μ and ν . This means that γ is not a Monge map since it sends every element in A to at least two locations, which contradicts Brenier's theorem. It follows that T is a global diffeomorphism between $\tilde{\Lambda}$ and $T(\tilde{\Lambda}) = \tilde{\Omega}'$.

Note that $\tilde{\Omega}' \subset \tilde{\Omega}$. Since T is optimal, $|\tilde{\Omega}'| = |T(\tilde{\Lambda})| = |\tilde{\Lambda}| = |\tilde{\Omega}|$ and therefore, $\tilde{\Omega}'$ is a set of full measure. This completes the proof of the lemma.

Finally, in our example below, we will need two additional properties, which we state now:

Property A: Restrictions of optimal maps are still optimal between the restricted domain and its image. [9, Theorem 4.6]

Property **B**: If the optimal map between Ω and Λ is of the form $\nabla \phi$, then the set $\{x \in \mathbb{R}^2 | \partial \phi(x) \cap \overline{\Lambda} \text{ contains a segment}\}$ is empty. [4, Proposition 3.2]

3.1. Explicit Example. We now introduce a specific example we refer to as the squareman. Let μ be the uniform probability measure on a rectangle Ω with sides

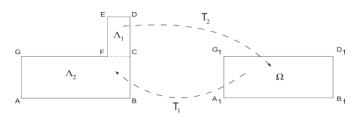


FIGURE 4. The squareman example. The optimal map T_1 pictured will be seen to be discontinuous.

 $|A_1B_1| = a$ and $|G_1A_1| = b$, and ν be the uniform probability measure on Λ , which is made up of two rectangles - a rectangle Λ_2 with sides |AB| = a and |GA| = b, and another rectangle Λ_1 on top of it with sides |CD| = c and |ED| = d, where d < a (see Figure 4). Note that since the optimal map is invariant under translations of Ω and Λ , it does not depend on the relative positions of the domains with respect to each other. Our example is similar to the one by Caffarelli: indeed, if we work with rectangles instead of disks, then the example by Caffarelli transforms to transporting a rectangle to an H-shape figure. Due to symmetry, we can divide these figures into 4 different symmetric parts and look at the OTM for each one. But this is exactly the example we are considering.

Since Ω is convex, it follows from Lemma 7 that the map T_2 is a diffeomorphism onto its image, which is a set of full measure in Ω . Further, by Proposition 11, it follows that there exists a continuous extension of T_2 from $\overline{\Lambda}$ to $\overline{\Omega}$. However, since Λ is not convex, we cannot make any such claims about T_1 . In fact, we will show below that T_1 is discontinuous, and further, give a qualitative statement of how discontinuous it is. Note that since $\partial \Lambda$ does not contain any connected subset with total signed curvature less than $-\pi/2$, this shows that the condition in Theorem 2 is not necessary.

Note that Brenier's theorem and Caffarelli's regularity theory are applicable only to the interiors of Ω and Λ . However, in this particular example, we will take advantage of the fact that the boundaries of the two domains are parts of straight segments. Using this, we will be able to identify where parts of the boundary $\partial \Lambda$ are mapped by T_2 , which will give us useful information about the discontinuity of the map $T_1 = \nabla \phi^*$. Indeed, if we knew that two points $x_1 \in EF$ and $x_2 \in GF$ are mapped to an interior point $x \in \Omega$, and since T_2 is continuous, then $\partial \phi^*(x)$ would contain both x_1 and x_2 , and in particular, the extent of discontinuity of T_1 at x would be at least the distance between x_1 and x_2 .

In the following four steps we will determine how the map T_2 behaves on the boundary of Λ and this will also give us information about the behaviour of T_1 . In particular, we will justify Figure 5, in which same-colored segments are mapped to same-colored segments. The 4 steps are as follows:

Step 1: $T_2(AB) = A_1B_1$ and $T_2(BD) = B_1D_1$ and both restrictions are homeomorphisms. Furthermore, $T_2(AG) \subset A_1G_1$ and $T_2(DE) \subset D_1G_1$.

Step 2: We have that either $T_2(G) = G_1$ or $T_2(E) = G_1$. In particular, we can assume without any loss of generality that $T_2(G) = G_1$, so $T_2(AG) = A_1G_1$ homeomorphically.

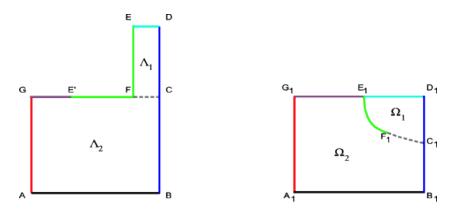


FIGURE 5. How the OTM behaves on the boundary of the domains.

We will further show that $T_2(EF)$ lies completely in Ω except for, of course, $T_2(E) = E_1$ which is on the boundary.

Step 3: For some $E' \subset GF$ we have that $T_2(GE') = G_1E_1$ homeomorphically. Step 4: $T_2(E'F) = T_2(EF)$.

Step 1: $T_2(AB) = A_1B_1$ and $T_2(BD) = B_1D_1$ and both restrictions are homeomorphisms. Furthermore, $T_2(AG) \subset A_1G_1$ and $T_2(DE) \subset D_1G_1$.

To get the desired information on $\partial \Lambda$, we are going to use what will henceforth be referred to as the reflection principle. More precisely, we reflect Λ with respect to ABto get a domain of twice the area, $R_{AB}(\Lambda)$ and reflect Ω with repect to A_1B_1 to get a domain $R_{A_1B_1}(\Omega)$. See Figure 6.

Let μ' and ν' be uniform probability measures on $R_{A_1B_1}(\Omega)$ and $R_{AB}(\Lambda)$ respectively with respective uniform densities f' and g'. As before, since $R_{A_1B_1}(\Omega)$ is convex, it follows from Caffarelli's regularity theory that the (ν -almost everywhere) unique, optimal map T'_2 from $R_{AB}(\Lambda)$ to $R_{A_1B_1}(\Omega)$ is smooth. We claim $T'_2(\Lambda) \subset \Omega$.

For simplicity let AB and A_1B_1 lie on the x-axis and for any $z \in \mathbb{R}^2$ let \overline{z} denote its reflection with respect to the x-axis. Due to symmetry, the map $T(x) = \overline{T'_2(\overline{x})}$ has the same cost as T'_2 and by the uniqueness of the OTM, we get $T'_2(x) = \overline{T'_2(\overline{x})}$ for a.e. x. But now if $T'_2(x) = y$ and x and y are on different sides of the x-axis, we know $T'_2(\overline{x}) = \overline{y}$. It is immediately checked that this violates the cyclic monotonicity condition for x and \overline{x} . In particular for almost all $x \in \Lambda$ we get that $T'_2(x) \in \Omega$ and since T'_2 is smooth we conclude that $T'_2(\Lambda) \subset \Omega$. Also note $T'_2(AB) \subset A_1B_1$ since otherwise by the continuity of T'_2 we could find a ball around a point on AB whose image under T'_2 lies completely above or below the x-axis, which as we explained above violates cyclic monotonicity. Note that there are two key conditions in the use of the reflection principle. First we need to reflect along a straight segment. Second we need one of the domains after reflection to be convex.

From here, since optimality is inherited by restriction (Property A), and since Brenier's theorem also guarantees almost-everywhere uniqueness of the optimal map, $T'_2|_{\Lambda}$

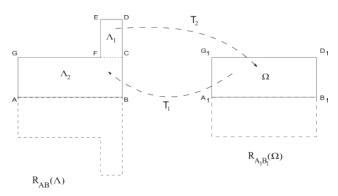


FIGURE 6. The *reflection principle*: We reflect both domains as to make the boundary part of the interior and use the optimal map is preserved under restriction.

coincides with T_2 almost everywhere. Since T'_2 is smooth, this gives us a smooth extension of T_2 to the interior of the segment AB and $T_2(AB) \subset A_1B_1$. Note that Proposition 11 already gives us a continuous extension of T_2 to the entire boundary $\partial \Lambda$. However, by using the reflection principle here, we have a smooth extension of T_2 to the interior of the segment AB, and more importantly, we get information about the image of this extension on the interior of the segment AB. Similarly, we get an extension of T_2 to the interior of BD and $T_2(BD) \subset B_1D_1$. We will use the same notation for T_2 and its continuous extensions to (parts of) $\partial \Lambda$.

Further, observe we can also use the reflection principle again and reflect $R_{AB}(\Lambda)$ with respect to the line containing BD to get $R'_{BD}(\Lambda)$ and reflect $R_{A_1B_1}(\Omega)$ with respect to the line containing B_1D_1 to get $R'_{B_1D_1}(\Omega)$. Now the newly obtained domains are 4 times the size of the original ones and $R'_{B_1D_1}(\Omega)$ is still convex. The motivation for doing so is to include the points B and B_1 in the interiors of the domains $R'_{BD}(\Lambda)$ and $R'_{B_1D_1}(\Omega)$ respectively. Exactly as above due to symmetry and smoothness of the optimal map, it follows that the optimal map must send B to B_1 . Therefore using Lemma 7 we have that T_2 maps the half-open segment BA (with B included) injectively to a (possibly strict) subset of B_1A_1 (with B mapping to B_1)

Similarly, we can reflect $R_{AB}(\Lambda)$ with respect to the line containing AG to get $R''_{AG}(\Lambda)$ and reflect $R_{A_1B_1}(\Omega)$ with respect to the line containing A_1G_1 to get $R''_{A_1G_1}(\Omega)$. Using exactly the same arguments as in the preceding paragraph, it follows that T_2 extends to a continuous, injective map on the entire closed segment AB, $T_2(A) = A_1, T_2(B) = B_1$ and $T_2(AB) \subset A_1B_1$. In fact, since $T_2(A) = A_1$ and $T_2(B) = B_1$, we have that $T_2(AB) = A_1B_1$ so that T_2 is a homeomorphism between the closed segments AB and A_1B_1 .

Note that symmetry considerations in the optimal map from $R''_{AG}(\Lambda)$ to $R''_{A_1G_1}(\Omega)$ lead to the conclusion that the interior of the segment AG must be mapped to a part of the interior of the segment A_1G_1 . Since we cannot perform any reflection of the form considered earlier to include G in the interior of the reflected domain, we cannot claim that T_2 can be extended to a continuous map till G and, in particular, that $T_2(G) = G_1$. In fact, as we will end up showing $T_2(G)$ might be distinct from G_1 . It is clear now, using exactly the same arguments as above, that $T_2(D) = D_1$ and in fact, that T_2 is a homeomorphism between the closed segments BD and B_1D_1 . Similarly, it also follows that T_2 extends smoothly to the interior of the segment DE and sends it to part of the interior of the segment D_1G_1 .

Step 2: We have that either $T_2(G) = G_1$ or $T_2(E) = G_1$.

After proving Step 2, we may thus assume (without loss of generality) that $T_2(G) = G_1$, so $T_2(AG) = A_1G_1$ homeomorphically. We will further show that $T_2(EF)$ lies completely in Ω except for, of course, $T_2(E) = E_1$ which is on the boundary.

To check the above first note that $T_2(\overline{\Lambda}) \subset \overline{\Omega}$ is of full measure and is compact, so it must be that $T_2(\overline{\Lambda}) = \overline{\Omega}$. Furthermore we know $T_2(\Lambda) \subset \Omega$, so it must be that $\partial \Omega \subset T_2(\partial \Lambda)$. Since in Step 1 we determined where $\partial \Lambda$ is mapped except for the segments GF and EF we must have that

(7)
$$G_1 \in T_2(GF)$$
 or $G_1 \in T_2(EF)$.

Now we will study $T_2(EF)$. Set $\alpha = T_2(EF \cup FC)$. Note α is a continuous curve from E_1 to $T_2(C) = C_1$. Then $\alpha \cup C_1D_1 \cup D_1E_1$ is a continuous loop. Note it is simple since if a point x is an intersection point, then $\partial \phi^*(x)$ would contain a segment in $\overline{\Lambda_1} \subset \overline{\Lambda}$, which contradicts (Property **B**). Hence by the Jordan curve theorem the loop bounds some open set $\Omega_1 \subset \Omega$, and we will refer to this loop as $\partial \Omega_1$. Note that neither $T_2(\Lambda_2)$ nor $T_2(\Lambda_1)$ can intersect $\partial \Omega_1 \cap \Omega$ (Property **B**). Also, both $T_2(\Lambda_2)$ and $T_2(\Lambda_1)$ are path connected, since Λ_1 and Λ_2 are path connected and T_2 is continuous. In particular, $T_2(\Lambda_1)$ is either completely contained inside Ω_1 or completely contained inside $\Omega \setminus \overline{\Omega_1}$, and a similar statement also holds for $T_2(\Lambda_2)$. Finally, note that by Step 1, the intersection of a ball of sufficiently small radius centered at D_1 with $\overline{\Omega}$ is contained inside $\overline{\Omega_1}$.

We claim that $T_2(\Lambda_1) \subset \Omega_1$. Indeed, suppose this is not so. Then, we must have $T_2(\Lambda_1)$ is contained in $\Omega \setminus \overline{\Omega_1}$ as noted above. We will show that this leads to a contradiction. To see this, choose a continuous path γ in Λ_1 with one endpoint at D. Since $T_2(D) = D_1$, and the intersection of a sufficiently small ball centered at D_1 with $\overline{\Omega}$ is contained inside $\overline{\Omega_1}$, it follows that there exist points in γ which must be mapped to Ω_1 by T_2 , so that we cannot have $T_2(\Lambda_1) \subset \Omega \setminus \overline{\Omega_1}$. Therefore, $T_2(\Lambda_1) \subset \Omega_1$ and in fact, $T_2(\Lambda_1)$ is of full measure in Ω_1 since $T_2(\Lambda) \subset \Omega$ is of full measure in Ω . Since the restriction of an optimal map (Property **A**) is optimal we get that $T_{2|\Lambda_1} \colon \Lambda_1 \to \Omega_1$ is the optimal map between the two domains.

Now we are in a position to study $T_2(EF)$. Let $T_2(K) = K_1$ be the point on $T_2(EF)$ satisfying the following two properties: (a) it lies on G_1E_1 (b) it has the least distance to G_1 among all the points on EF whose images under T_2 lies on G_1E_1 . We will show that $K_1 = E_1$.

Observe first that $K_1E_1 \subset T_2(EF)$. Indeed if this was not the case then for some $M \in EK$ we would have that $T_2(M) \in \Omega$, but then the pair of points K, M would violate the cyclical monotonicity condition. Note that even though these points are on the boundary of Λ , the continuity of T_2 allows us to extend the cyclical monotonicity condition to these points. Hence $K_1E_1 \subset T_2(EF)$.

Now, we reflect Ω_1 with respect to K_1D_1 and reflect Λ_1 with respect to ED. The reflection of Λ_1 is a convex domain, and therefore, we can use the reflection principle as before, from which we can conclude that $T_1(K_1D_1) \subset ED$. Note here we are using the fact that $\overline{\Lambda_1}$ is convex to be able to define T_1 everywhere on $\overline{\Omega_1}$ (Proposition 11). However, we already knew that $T_2(ED) = E_1D_1$, so we conclude $K_1D_1 \subset E_1D_1$, so $K_1 = E_1$ as desired.

In particular, we get that $T_2(EF) \cap E_1G_1 = E_1$. Applying the same argument for $T_2(GF)$ we conclude that

(8)
$$T_2(EF) \cap E_1G_1 = E_1 \text{ and } T_2(GF) \cap \overline{G_1}G_1 = \overline{G_1}$$

where $\overline{G_1} = T_2(G)$. In particular, (7) and (8) imply that $T_2(E) = E_1 = G_1$ or $T_2(G) = G_1$. Due to symmetry between E and G, we can assume without loss of generality that $T_2(G) = G_1$. Further, if $x \in T_2(EF) \cap (\partial \Omega \setminus G_1 E_1)$ then since we have covered $\partial \Omega \setminus G_1 E_1$ by boundary segments of $\partial \Lambda \setminus EF \cup GF$ we would get that $\partial \phi^*(x)$ contains a segment in $\overline{\Lambda}$, which contradicts (Property **B**). Hence $T_2(EF)$ is in Ω except for $T_2(E) = E_1 \in \partial \Omega$. Note that here we used the fact that the only points of Λ_1 that can be mapped to the same point are pairs of points on EF and GF since the segment connecting them intersects $\overline{\Lambda}$ at isolated points and not segments (Property **B**)

Step 3: For some $E' \subset GF$ we have that $T_2(GE') = G_1E_1$ homeomorphically.

As we explained at the beginning of Step 2 we must have that $\partial \Omega \subset T_2(\partial \Lambda)$. The only part of $\partial \Lambda$ whose position under T_2 we haven't determined yet is GF. At the same time we know $G_1E_1 \cap T_2(\partial \Lambda \setminus GF) = \emptyset$, so we conclude that $G_1E_1 \subset T_2(GF)$. Let $E' \in GF$ be such that $T_2(E') = E_1$. It is clear now that by Property **B**, $T_2(GE')$ is simple. Further, note that if some $M \in E'F$ maps to the G_1E_1 then the pair M, E' would violate the cyclical monotonicity condition. Hence since $G_1E_1 \subset T_2(GF) = T_2(GE' \cup E'F)$ we must have that $G_1E_1 \subset T_2(GE')$. But now $T_2(GE')$ is simple with endpoints G_1 and E_1 and it contains G_1E_1 , so it must be that $T_2(GE') = G_1E_1$ as desired.

Step 4:
$$T_2(E'F) = T_2(EF)$$

Set $\alpha_1 = T_2(EF)$ and $\alpha_2 = T_2(FC)$ and note that both are simple curves (Property **B**). Following the notation from above we have that $\alpha = T_2(EF \cup FC) = \alpha_1 \cup \alpha_2$. Now $\alpha \cup C_1B_1 \cup B_1A_1 \cup A_1G_1 \cup G_1E_1$ is a Jordan curve, so it bounds some Ω_2 . In particular we have that $\Omega_1 \cup \Omega_2 \cup \mathring{\alpha} = \Omega$ where $\mathring{\alpha}$ is the curve without its endpoints. In Step 2 we showed that $T_2(\Lambda_1)$ is of full measure in Ω_1 . Hence $T_2(\Lambda_2) \subset \Omega_2$ and is of full measure in Ω_2 . Therefore $T_{2|\Lambda_2} : \Lambda_2 \to \Omega_2$ restricts to an optimal map by Property **A**. $T_{1|\Omega_2}$ is its inverse, so it is optimal as well.

But now Λ_2 is convex, so $T_{1|\Omega_2}$ is smooth on Ω_2 by Caffarelli's regularity theory and extends continuously to the boundary. Now $T_{1|\Omega_2}(\overline{\Omega_2}) \subset \overline{\Lambda_2}$ is compact and of full measure, so $T_1(\overline{\Omega_2}) = \overline{\Lambda_2}$. Since $T_1(\Omega_2) \subset \Lambda_2$ we conclude that $\partial \Lambda_2 \subset T_1(\partial \Omega_2)$. We know that T_2 sends $\partial \Lambda_2 \setminus E'F$ to $\partial \Omega_2 \setminus \alpha_1$ homeomorphically by the previous three steps. This means $T_1(\partial \Omega_2 \setminus \alpha_1) = \partial \Lambda_2 \setminus E'F$, so we must have that $E'F \subset T_1(\alpha_1)$. But note $T_1(\alpha_1)$ is a simple curve (by Property **B**) that contains E'F and has endpoints E' 18

and F. Hence it must be that $T_1(\alpha_1) = E'F$ homeomorphically, so we conclude that $T_2(E'F) = \alpha_1 = T_2(EF)$ as desired.

Thus we have completely determined $T_2(\partial \Lambda)$ and thus we have justified the picture in Figure 5. Further, we have determined that the set of discontinuity of T_1 is exactly the curve α_1 and for every $x \in \alpha_1$, the subdifferential $\partial \phi^*(x)$ is a segment that connects the preimages of x on E'F and EF. In particular, as x moves from $T_1(F)$ to E_1 this segment grows and reaches |EE'| as x gets to E_1 .

Appendix A.

Proposition 8. Assume Λ is a bounded simply connected domain with C^2 boundary. Then there exists an $\epsilon_o > 0$ such that for all $0 < \epsilon < \epsilon_o$, $\Gamma_{\epsilon} = \{x \in \Lambda : dist(x, \partial \Lambda) = \epsilon\}$ is a C^1 simple curve with a diffeomorphism $f_{\epsilon} : \partial \Lambda \to \Gamma_{\epsilon}$ such that $f_{\epsilon}(x) - x$ is normal to both curves at the points x and $f_{\epsilon}(x)$ and $|f_{\epsilon}(x) - x| = \epsilon$.

Proof. Define $v : \partial \Lambda \to S^1$ to be the inward pointing unit normal vector field. Next define $F : \partial \Lambda \to \Lambda$ by $F(x) = x + \epsilon v(x)$. This map is well defined for ϵ small, so that $Im(F) \subset \Lambda$. In fact we will show F is the desired diffeomorphic map f_{ϵ} .

Note that for any $x \in \partial \Lambda$ there exists $\epsilon_x = \sup\{\overline{\epsilon} > 0 : B_{\overline{\epsilon}}(x + \overline{\epsilon}v(x)) \cap \partial \Lambda = \{x\}\}$. We claim that there exists an $\epsilon_o > 0$ such that $\epsilon_x \ge \epsilon_o$ for all $x \in \partial \Lambda$. Indeed, this is true because $\partial \Lambda$ is assumed to be C^2 , so that its radius of curvature is a continuous, strictly positive function on $\partial \Lambda$, which is assumed to be compact. By taking ϵ_o to be smaller than the positive lower bound for the radius of curvature, the proof of the claim is complete. Therefore, for $0 < \epsilon < \epsilon_o$, we get $\epsilon = |F(x) - x| = \operatorname{dist}(F(x), \partial \Lambda)$, for any $x \in \partial \Lambda$, so that in particular $Im(F) \subset \Gamma_{\epsilon}$. Furthermore if $y \in \Gamma_{\epsilon}$ then there exists $x \in \partial \Lambda$ with $|x - y| = \epsilon$, so in particular y = F(x). Hence $Im(F) = \Gamma_{\epsilon}$. Next note that F is clearly 1-1 since $\overline{B_{\epsilon}(F(x))} \cap \partial \Lambda = \{x\}$ for every $x \in \partial \Lambda$. Hence F is a bijection.

Finally, note that since $\partial \Lambda$ is C^2 then v is C^1 and so F is C^1 . Hence $Im(F) = \Gamma_{\epsilon}$ is a C^1 curve which is diffeomorphic to $\partial \Lambda$. Also by definition x - F(x) is normal to $\partial \Lambda$ at x and since dist $(x, \Gamma_{\epsilon}) = \epsilon = |x - F(x)|$ we also have that x - F(x) is normal to Γ_{ϵ} at F(x). This completes the proof of the proposition.

Proposition 9. Assume $\gamma : [0,1] \to \mathbb{R}^2$ is a C^2 simple (non-closed), regular curve. Then there exists $t \in (0,1)$ such that $\gamma'(t)$ is parallel to $\gamma(1) - \gamma(0)$.

Proof. Let $\gamma(s) = (\gamma_1(s), \gamma_2(s))$. By Cauchy's mean value theorem, there exists some $t \in (0, 1)$ such that $(\gamma_1(1) - \gamma_1(0))\gamma'_2(t) = (\gamma_2(1) - \gamma_2(0))\gamma'_1(t)$. Without loss of generality, we can assume that $\gamma_2(1) - \gamma_2(0) \neq 0$. Then, if $\gamma'_2(t) = 0$, we get that $\gamma'_1(t) = 0$, which contradicts the regularity assumption. Dividing by $(\gamma_2(1) - \gamma_2(0))\gamma'_2(t)$, we have the result.

Proposition 10. Assume Λ is a bounded simply connected domain with C^2 boundary. Let $\epsilon > 0$ be such that $\Gamma_{\epsilon} = \{x \in \Lambda : dist(x, \partial \Lambda) = \epsilon\}$ is a C^1 simple curve and $f_{\epsilon} : \partial \Lambda \to \Gamma_{\epsilon}$ be a diffeomorphism such that $f_{\epsilon}(x) - x$ is normal to both curves at the points x and $f_{\epsilon}(x)$ and $|f_{\epsilon}(x) - x| = \epsilon$ (guaranteed to exist by Proposition 8). Now let $\gamma : [0,1] \to \mathbb{R}^2$ be a simple C^1 closed curve that lies in Λ and contains Γ_{ϵ} in its interior in the sense that every continuous path between Γ_{ϵ} and $\partial \Lambda$ must intersect γ . Also assume that γ intersects neither $\partial \Lambda$ nor Γ_{ϵ} . Then for all pairs $C, D \in \partial \Lambda$ of distinct points, if $I \subset \partial \Lambda$ is the subset of $\partial \Lambda$ that connects C and D as we move clockwise around $\partial \Lambda$ there exists a connected subset of γ that lies completely in the closed figure Θ_{CD} bounded by the segments $[C, C_2], [D, D_2]$ and by I and $f_{\epsilon}(I)$ such that it has one endpoint on each of $[C, C_2]$ and $[D, D_2]$, where $C_2 = f_{\epsilon}(C)$ and $D_2 = f_{\epsilon}(D)$.

This proposition is illustrated in Figure 2, where the path between P and Q is the connected subset of γ which the proposition guarantees.

Proof. We prove this by contradiction. Without loss of generality, we can assume that $\gamma(0)$ lies outside Θ_{CD} . Let $t_1 = \inf\{t: \gamma(t) \in \Theta_{CD}\}$ and $t_2 = \sup\{t: \gamma(t) \in \Theta_{CD}\}$. By the compactness of the unit interval, the continuity of γ , and the closedness of Θ_{CD} , the infimum and supremum are attained.

Consider $S = \gamma[t_1, t_2] \cap \Theta_{CD}$. Since we are assuming that there does not exist a connected subset of γ that lies completely in the closed figure Θ_{CD} such that it has one endpoint on each of $[C, C_2]$ and $[D, D_2]$, it follows that there are two, mutually exclusive types of path connected components of S - those that intersect $[CC_2]$ and those that intersect $[DD_2]$. Let A' denote the union of all the path connected components of S that intersect $[DD_2]$. Then, A' and B' are disjoint, and we claim that they are also compact subsets of \mathbb{R}^2 .

First, let us see how the compactness of A' and B' finishes the proof. Let Δ'_{CD} denote the complement of Θ_{CD} in the region enclosed between $\partial \Lambda$ and Γ_{ϵ} . Let $\Delta_{CD} = \Delta'_{CD} \cup [CC_2] \cup [DD_2]$. Note that Δ_{CD} is a closed and bounded, hence compact subset of \mathbb{R}^2 . Finally, let $A = A' \cup \Delta_{CD}$ and $B = B' \cup \Delta_{CD}$. Then, the compactness of A', B'implies that A, B are compact and further, for E in the interior of I and $E_2 = f_{\epsilon}(E)$, we get that E, E_2 are not separated by A or B in the sense that they lie in the same path connected (and hence, connected) component of $\mathbb{R}^2 \setminus A$ and of $\mathbb{R}^2 \setminus B$. To see that E, E_2 are not separated by A, we begin by noting that the compactness of A' implies that the distance between A' and $[D, D_2]$ is always greater than some $\epsilon > 0$. Then, we can go from E, E_2 in $\mathbb{R}^2 \setminus A$ by travelling along I towards D until we are at a distance $\epsilon/2$ away from D, then moving parallel to $[D, D_2]$ until we hit $f_{\epsilon}(I)$, and finally, moving along $f_{\epsilon}(I)$ to E_2 . Here, we use the fact that A' does not intersect I or $f_{\epsilon}(I)$. A similar argument shows that E, E_2 are not separated by B.

Now, we recall Janiszewski's theorem [3, Ap.3.2], which says that if A, B are compact subsets of \mathbb{R}^2 such that $A \cap B$ is connected, and $E, E_2 \in \mathbb{R}^2 \setminus A \cup B$ such that neither Anor B separates E, E_2 , then $A \cup B$ also does not separate E, E_2 . This implies that γ does not separate E, E_2 i.e. E, E_2 lie in the same connected component of $\mathbb{R}^2 \setminus \gamma$ (and hence, the same path connected component of the open subset $\mathbb{R}^2 \setminus \gamma$ of \mathbb{R}^2). But this contradicts the hypothesis that Γ_{ϵ} lies in the interior of γ .

So, to finish the proof, we only need to show that A', B' are compact. We do this only for A', the proof for B' being similar. Since A' is a bounded subset of \mathbb{R}^2 , we only need to show that it is closed in \mathbb{R}^2 . Let p_n be a sequence of points in A' such that $p_n \to p$. There exists a unique sequence $\{a_n\} \in [t_1, t_2]$ such that $p_n = \gamma(a_n)$. By the compactness of $[t_1, t_2]$, we can, after possibly passing to a subsequence, assume that $a_n \to a \in [t_1, t_2]$. After possibly passing to another subsequence, we can further assume that either $a_n < a$ or $a_n > a$ (since the case where a_n is eventually a is trivial). In the subsequent discussion, we will assume that $a_n < a$. The case $a_n > a$ is treated similarly. By the continuity of γ , it follows that $p = \gamma(a)$. Since $a \in [t_1, t_2]$, we have that either $p \in A'$ or $p \in B'$. If $p \in A'$, then we are done, so suppose that $p \in B'$. Since $\gamma[a_n, a]$ is path connected but $\gamma(a_n)$ and $\gamma(a)$ are in different path connected components of S, it follows that $\gamma[a_n, a]$ must leave S through $[C, C_2]$. Let $a_n < b_n < a$ be such that $\gamma(b_n) \in [C, C_2]$. Note that we can always find such a b_n by the previous remark. But then, $b_n \to a$, so that from the continuity of γ and the closedness of $[C, C_2]$ we get $\gamma(a) = \lim_n \gamma(b_n) \in [C, C_2]$, which contradicts that $\gamma(a) \in B'$. This completes the proof.

Proposition 11. Let Ω and Λ be bounded, connected, simply connected open domains in \mathbb{R}^2 , equipped with the uniform measures μ and ν . Assume that $\overline{\Omega}$ convex. Then, the OTM T: $\Lambda \to \Omega$ admits a single-valued, continuous extension to $\overline{\Lambda}$.

Proof. By considering ν as a measure on all of \mathbb{R}^2 supported on Λ , Brenier's theorem furnishes a globally Lipschitz convex function $\varphi \colon \mathbb{R}^2 \to \mathbb{R}$ such that $T = \nabla \varphi$ on Λ (the equality holds everywhere, instead of just almost everywhere, because of Caffarelli's regularity theory), and $\partial \varphi(\mathbb{R}^2) \subset \overline{\Omega}$ since $\overline{\Omega}$ is convex [1, Lemma 1(b)]. For $x \in \mathbb{R}^2$, the set $\partial \varphi(x)$ is convex [6, p. 215], and so if it is not a singleton it contains a segment that is contained in the convex set $\overline{\Omega}$. However, this contradicts Property **B** stated in §3. In particular, φ is differentiable on \mathbb{R}^2 and therefore is C^1 [6, Theorem 25.5], implying the statement.

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